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Some Remarks about the Collocation Method on a Modified Legendre Grid

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Abstract—We compare the results obtained by applying the standard collocation method at the Legendre Gauss-Lobatto nodes, for a model problem simulating a steady advection-diffusion equation, with those obtained by collocating at a new set of nodes. These nodes are derived from a suitable modification of the Legendre grid, according to the magnitude of the ratio between the advective and the diffusive parts of the differential operator.

Keywords—Legendre polynomials, Collocation method, Boundary layer.

1. INTRODUCTION

In computing approximated solutions of advection-diffusion boundary-value problems, where the advective part dominates the diffusive part, spurious oscillations may occur when the degrees of freedom are not sufficient to recover a satisfactory resolution of the boundary layers. This is also true when the approximated solutions are algebraic polynomials determined by using the collocation method at the grid of the Legendre Gauss-Lobatto nodes (or other set of nodes of Jacobi type). For a survey of these approximation techniques, known as spectral methods, we refer, for instance, to [1–4]. Following an idea proposed in [5], we consider approximated polynomial solutions obtained by implementing the collocation method on a special grid. The nodes of this new grid are the zeroes of a linear combination of the Legendre polynomial P_n and its derivative, and the coefficients of the linear combination are related to the coefficients of the differential operator in the equation, in order to take into consideration the dominance of the advective terms on the diffusive terms. This amounts to modify the Legendre grid by moving the points in the upwind direction with respect to the local flux, of a quantity depending on the magnitude of the flux itself. No modification is made when the differential operator only contains second-order partial derivatives (i.e., when it is symmetric), so that for pure diffusive equations the new grid coincides with the usual Legendre grid.

One of the main advantages in using the modified grid is that the corresponding approximated solutions do not oscillate as much as the ones resulting from the use of classical grids, providing more reliable and accurate representations of the exact solution. Some comparisons have been presented in [6]. Unfortunately, a theory of stability and convergence has not been developed yet. Presented in this paper are some preliminary materials and some conjectures with the aim of better understanding the principles of the method and the reasons of its success. Results and observations are carried out for a very simple steady equation in one dimension, but the method has been numerically tested on more serious examples in [6].

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2. THE CLASSICAL APPROACH

To introduce our approximation techniques, we work with a model equation, i.e., we are concerned with finding the solution u of the following boundary value problem:

$$\begin{aligned} -\epsilon u'' + u' &= 0, \qquad \text{in }]-1,1[, \\ u(-1) &= 0, \\ u(1) &= 1, \end{aligned} \tag{2.1}$$

where $\epsilon > 0$ is given. The solution is explicitly described by the expression

$$u(x) = \frac{e^{x/\epsilon} - e^{-1/\epsilon}}{e^{1/\epsilon} - e^{-1/\epsilon}}, \qquad x \in [-1,1], \tag{2.2}$$

and presents a boundary layer at the point $x = 1$ as shown in Figure 1 for $\epsilon = 1/20$.

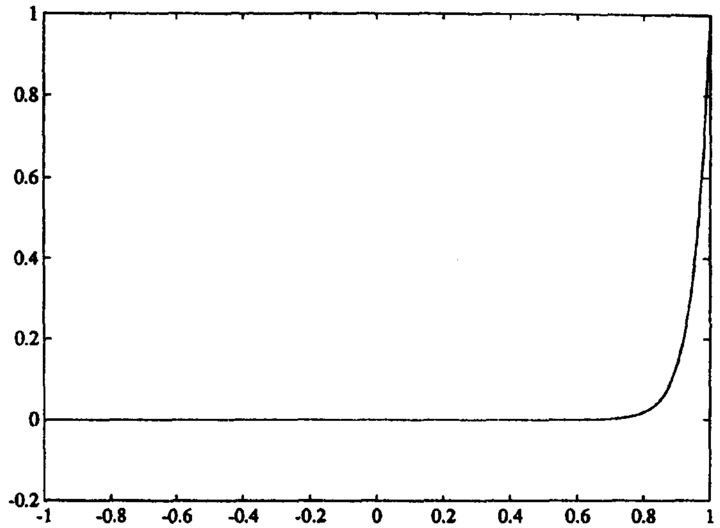

Figure 1. Solution to (2.1) for $\epsilon = 1/20$.

We would like to approximate u by algebraic polynomials using the collocation method at the Legendre points. To this purpose, for any $n \geq 1$, let us define $\mathbf{P}_n$ as the space of polynomials of degree less than or equal to n. Then, we define the Gauss nodes $\xi_i^{(n)}$, $1 \leq i \leq n$, to be the zeroes of the Legendre polynomial P_n, and the Gauss-Lobatto nodes $\eta_i^{(n)}$, $0 \leq i \leq n$, to be the zeroes of the polynomial $(1-x^2)P_n'$. Both the sets of nodes will be ordered increasingly, so that $\eta_0^{(n)} = -1$ and $\eta_n^{(n)} = 1$.

The approximated solution $q_n \in \mathbf{P}_n$ is required to satisfy the set of equations

$$\begin{aligned} -\epsilon q_n''\left(\eta_i^{(n)}\right) + q_n'\left(\eta_i^{(n)}\right) &= 0, \qquad 1 \leq i \leq n-1, \\ q_n(-1) &= 0, \\ q_n(1) &= 1; \end{aligned} \tag{2.3}$$

that is, we collocated the differential equation in (2.1) at the internal Gauss-Lobatto points and we imposed the boundary conditions. By writing q_n in terms of the Lagrange polynomials $l_j^{(n)} \in \mathbf{P}_n$, $0 \leq j \leq n$, with respect to the nodes $\eta_i^{(n)}$, $0 \leq i \leq n$, we have

$$q_n(x) = \sum_{j=0}^{n} q_n\left(\eta_j^{(n)}\right) l_j^{(n)}(x), \qquad \text{where } l_j^{(n)}\left(\eta_i^{(n)}\right) = \delta_{ij}. \tag{2.4}$$

More precisely, one has

$$l_j^{(n)}(x) = \frac{1}{n(n+1)} \times \begin{cases} -(-1)^n(1-x)P_n'(x), & \text{if } j=0, \\ \dfrac{-\left(1-x^2\right)P_n'(x)}{P_n\left(\eta_j^{(n)}\right)\left(x-\eta_j^{(n)}\right)}, & \text{if } 1 \le j \le n-1, \\ (1+x)P_n'(x), & \text{if } j=n. \end{cases} \tag{2.5}$$

Thus, an equivalent expression of (2.3) is

$$\begin{aligned} \sum_{j=0}^{n}\left[-\epsilon\left(\frac{d^2}{dx^2}l_j^{(n)}\right)\left(\eta_i^{(n)}\right)+\left(\frac{d}{dx}l_j^{(n)}\right)\left(\eta_i^{(n)}\right)\right]q_n\left(\eta_j^{(n)}\right) &= 0, \qquad 1 \le i \le n-1, \\ q_n(-1) &= 0, \\ q_n(1) &= 1, \end{aligned} \tag{2.6}$$

which shows that the collocation problem is equivalent to a linear system in the unknowns $q_n(\eta_j^{(n)})$, $0 \le j \le n$. The entries of the matrix corresponding to such a linear system can be computed by observing that (see, for instance, [2])

$$d_{ij} = \left(\frac{d}{dx}l_j^{(n)}\right)\left(\eta_i^{(n)}\right) = \begin{cases} -\dfrac{1}{4}n(n+1), & \text{if } i=j=0, \\ \dfrac{P_n\left(\eta_i^{(n)}\right)}{P_n\left(\eta_j^{(n)}\right)}\dfrac{1}{\eta_i^{(n)}-\eta_j^{(n)}}, & \text{if } 0 \le i \le n, \quad 0 \le j \le n, \quad i \ne j, \\ 0, & \text{if } 1 \le i = j \le n-1, \\ \dfrac{1}{4}n(n+1), & \text{if } i=j=n. \end{cases} \tag{2.7}$$

The entries corresponding to the second derivative operator $(\frac{d^2}{dx^2}l_j^{(n)})(\eta_i^{(n)})$ can be computed by squaring the $(n+1)\times(n+1)$ matrix $\{d_{ij}\}_{\substack{0\le i\le n \\ 0\le j\le n}}$ in (2.7).

Without discussing the details of the proof, we just mention that one can show that (2.6) admits a unique solution. Moreover, for $n \to +\infty$, q_n converges to u with a rate depending on the regularity of u itself. These well-known results are discussed, for instance, in [1–4]. In particular, when n is sufficiently large (that means n of the order of $1/\sqrt{\epsilon}$) we have enough degrees of freedom to accurately resolve the boundary layer. In this case, the approximated solutions are monotone and they start to converge with an exponential rate to the analytic solution u in (2.2). On the other hand, if ϵ is very small, as may be required in many practical applications, we would like to have information on the behavior of the solution also when n is not too large. Unfortunately, if n is small compared to $1/\sqrt{\epsilon}$, the polynomials q_n are polluted by oscillations as illustrated in Figure 2. We note that, for fixed n and a decreasing ϵ, q_n increases its oscillations until it blows up for ϵ tending to zero. This phenomenon is also observed for centered finite-differences approximations (see [6]).

Our aim is to modify the collocation nodes in relation to the magnitude of ϵ in order to recover acceptable approximated solutions even when $n \ll 1/\sqrt{\epsilon}$.

3. THE NEW APPROACH

We define a new set of nodes $\tau_j^{(n)}$, $1 \le j \le n$. They are the n distinct zeroes of a suitable linear combination of the Legendre polynomial P_n and its derivative. More precisely, we require that

$$-\epsilon P_n'\left(\tau_j^{(n)}\right) + P\left(\tau_j^{(n)}\right) = 0, \qquad 1 \le j \le n. \tag{3.1}$$

It is not difficult to prove the following relation:

$$\xi_j^{(n)} < \tau_j^{(n)} < \eta_j^{(n)}, \qquad \text{for } 1 \le j \le n-1. \tag{3.2}$$

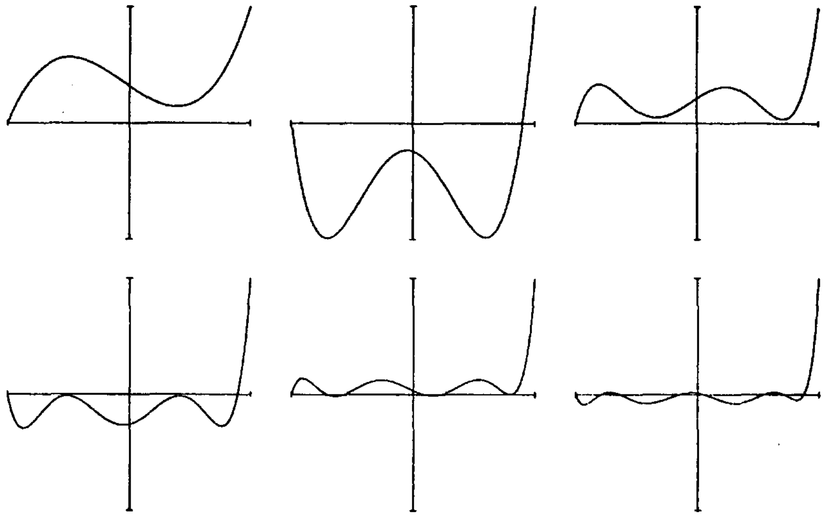

Figure 2. The polynomials q_n for $\epsilon = 1/20$ and $3 \leq n \leq 8$.

We also set $\tau_j^{(n)} = \xi_j^{(n)}$, $1 \leq j \leq n$, when $\epsilon = 0$ and $\tau_j^{(n)} = \eta_j^{(n)}$, $1 \leq j \leq n-1$, when $\epsilon = +\infty$. The reason of this new choice of nodes is explained in [5]. Here, we only observe that the collocation of the equation (2.1) at the nodes $\tau_j^{(n)}$, $1 \leq j \leq n-1$, brings to approximated solutions which behave much better than those obtained with the classical approach, also when n is small in comparison to $1/\sqrt{\epsilon}$. Therefore, we now look for $s_n \in \mathbf{P}_n$ such that

$$\begin{aligned} -\epsilon s_n''\left(\tau_i^{(n)}\right) + s_n'\left(\tau_i^{(n)}\right) &= 0, \qquad 1 \leq i \leq n-1, \\ s_n(-1) &= 0, \\ s_n(1) &= 1. \end{aligned} \tag{3.3}$$

Note that the node $\tau_n^{(n)}$ is not used. It is not difficult to write the linear system corresponding to (3.3) by differentiating the Lagrange polynomials in (2.5) and evaluating at the collocation nodes. We plot in Figure 3 some of the polynomials s_n when $\epsilon = 1/20$. The improvement is evident. Such a situation was also pointed out in [6] for finite-differences discretizations using the same type of modified nodes. We will try to explain why the new set of nodes provides such good results.

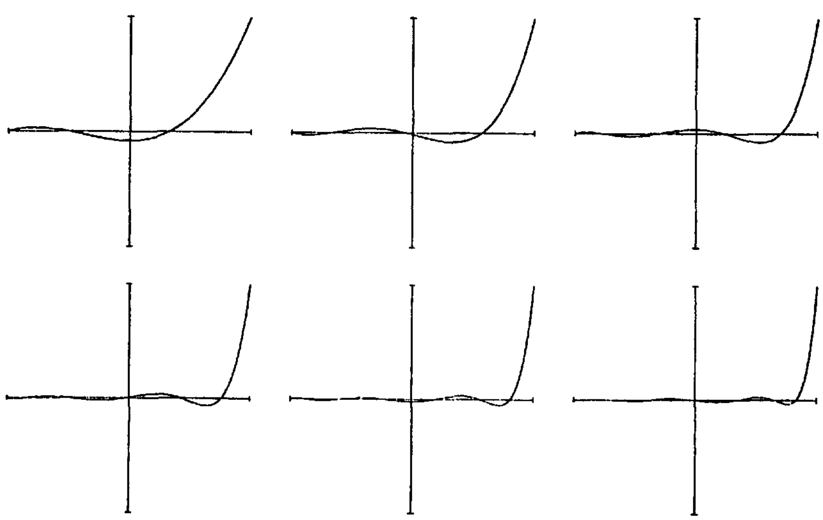

Figure 3. The polynomials s_n for $\epsilon = 1/20$ and $3 \leq n \leq 8$.

Given $\mu \geq 0$, we start by defining the polynomial $Q_{n,\mu} \in \mathbf{P}_{n-1}$ such that

$$Q_{n,\mu}(x) = \begin{cases} \dfrac{-\mu P_n'(x) + P_n(x)}{x - \tau_{n,\mu}^{(n)}}, & \text{if } \mu \geq 0, \\ \dfrac{1}{n} P_n'(x), & \text{if } \mu = +\infty, \end{cases} \tag{3.4}$$

where $\tau_{n,\mu}^{(n)}$ is the largest zero of $-\mu P_n' + P_n$ $(\mu \neq +\infty)$. Note that $\tau_{n,\epsilon}^{(n)} = \tau_n^{(n)}$ and $\tau_{n,0}^{(n)} = \xi_n^{(n)}$. We also note that

$$Q_{n,\mu} - Q_{n,\infty} \in \mathbf{P}_{n-2}, \qquad \forall \mu \geq 0, \tag{3.5}$$

which says that $Q_{n,\mu}$ and $Q_{n,\infty}$ have the same coefficient of the monomial x^{n-1}.

At this point, since $-\epsilon s_n'' + s_n' \in \mathbf{P}_{n-1}$, problem (3.3) may be written as

$$\begin{aligned} -\epsilon s_n''(x) + s_n'(x) &= \alpha_{n,\epsilon}\, Q_{n,\epsilon}(x), \qquad \forall x \in \left]-1,1\right[, \\ s_n(-1) &= 0, \\ s_n(1) &= 1, \end{aligned} \tag{3.6}$$

where $\alpha_{n,\epsilon}$ is a suitable constant to be determined using the boundary conditions (see Section 5).

4. BEHAVIOR OF THE NEW COLLOCATION NODES

Before continuing, we make some considerations regarding the distribution of the nodes $\tau_j^{(n)}$, $1 \leq j \leq n$. If $\mu \neq +\infty$ we first analyze the point $\tau_{n,\mu}^{(n)}$. This satisfies

$$\tau_{n,\mu}^{(n)} > 1, \qquad \text{if and only if } n > \frac{1}{2}\left(-1 + \sqrt{1 + \frac{8}{\mu}}\right). \tag{4.1}$$

In fact, it is sufficient to observe that $\tau_{n,\mu}^{(n)} = 1$ if and only if $\mu P_n'(1) = P_n(1)$, which is realized when n is the root of the equation $n^2 + n - 2/\mu = 0$ (recall that $P_n(1) = 1$ and $P_n'(1) = n(n+1)/2$. The behavior of the point $\tau_{n,\mu}^{(n)}$ is shown for different values of the parameters n and μ in Figure 4 (n integer). Note that $\lim_{n \to +\infty} \tau_{n,\mu}^{(n)} = +\infty$.

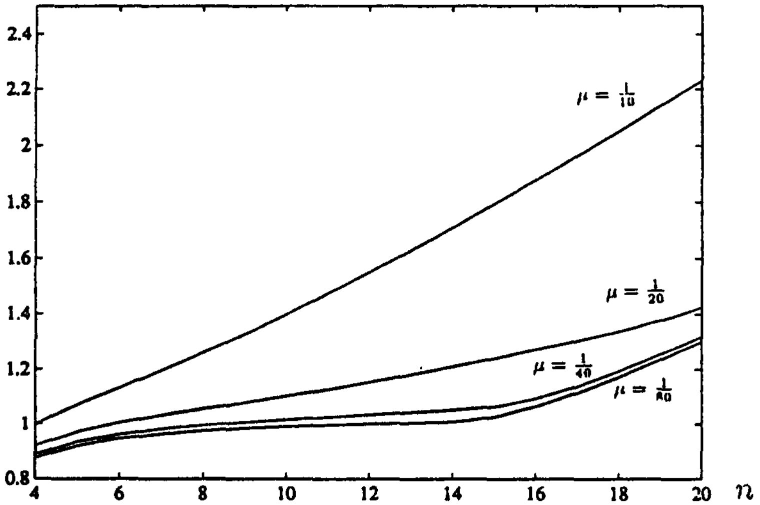

Figure 4. The point $\tau_{n,\mu}^{(n)}$ for different n and μ.

Concerning the other nodes, we begin by observing that (see [7])

$$\xi_j^{(n)} \approx -\cos\left(\frac{2j-1/2}{2n+1}\pi\right), \qquad 1 \le j \le n, \tag{4.2}$$

$$\eta_j^{(n)} \approx -\cos\left(\frac{2j+1/2}{2n+1}\pi\right), \qquad 1 \le j \le n-1. \tag{4.3}$$

With the help of some trigonometry we may arrive to conjecture that

$$\tau_j^{(n)} \approx -\cos\left(\frac{2j-1/2}{2n+1}\pi + \frac{2}{2n+1}\arccos\frac{2}{\sqrt{4+(2n+1)^2\epsilon^2}}\right), \qquad 1 \le j \le n-1. \tag{4.4}$$

The above estimate should demonstrate that when $\epsilon \to +\infty$ or $n \to +\infty$ then $\tau_j^{(n)} \to \eta_j^{(n)}$, while for $\epsilon = 0$ one has $\tau_j^{(n)} = \xi_j^{(n)}$.

Another way to determine an approximated value of $\tau_j^{(n)}$ is to intersect the straight-line generating from $\xi_j^{(n)}$ and tangent to P_n with the straight-line generating from $\eta_j^{(n)}$ and tangent to $\epsilon P_n'$. This implies

$$\tau_j^{(n)} \approx \frac{\epsilon n(n+1)\eta_j^{(n)} P_n\left(\eta_j^{(n)}\right) - \xi_j^{(n)} P_n'\left(\xi_j^{(n)}\right)\left(1-\left[\eta_j^{(n)}\right]^2\right)}{\epsilon n(n+1) P_n(\eta_j^{(n)}) - P_n'\left(\xi_j^{(n)}\right)\left(1-\left[\eta_j^{(n)}\right]^2\right)}, \qquad 1 \le j \le n-1, \tag{4.5}$$

where we have used that $P_n''(\eta_j^{(n)}) = -(1-[\eta_j^{(n)}]^2)^{-1} n(n+1) P_n(\eta_j^{(n)})$, $1 \le j \le n-1$.

When $\epsilon > 0$ is fixed and n tends to infinity, we deduce from (3.2) that $\tau_j^{(n)}$ converges to $\eta_j^{(n)}$, since the distance $|\eta_j^{(n)} - \xi_j^{(n)}|$ behaves like $1/n$ at the center of the interval $[-1,1]$, and like $1/n^2$ near the endpoints. Using (4.4), we may say more about the rate of convergence of $\tau_j^{(n)}$ to $\eta_j^{(n)}$. For $n \gg 1/\sqrt{\epsilon}$, we can derive the asymptotic estimate

$$\left|\eta_j^{(n)} - \tau_j^{(n)}\right| \approx \frac{1}{\epsilon n^2}\sqrt{1-\left[\eta_j^{(n)}\right]^2}, \qquad 1 \le j \le n-1. \tag{4.6}$$

This would say that, for $n \to +\infty$, the distance $|\eta_j^{(n)} - \tau_j^{(n)}|$ tends to zero faster than the difference $|\eta_j^{(n)} - \xi_j^{(n)}|$. In practice, the two polynomials q_n and s_n, solutions, respectively, to equations (2.3) and (3.3), do not differ too much as n is large. Heuristically, this would also prove that s_n converges to u with an exponential rate. Unfortunately, we do not have the rigorous proof of these statements. On the other hand, the numerical experiments not only confirm that s_n converges to u, but they also show, as we deduce by comparing Figures 2 and 3, that s_n is accurate also if n is not large. We need to investigate further to make more clear this fact.

5. COMPARISON BETWEEN THE TWO APPROACHES

Integrating the differential equation in (3.6) we get

$$-\epsilon\, s_n'(x) + s_n(x) = \beta_{n,\epsilon} + \alpha_{n,\epsilon}\int_{-1}^{x} Q_{n,\epsilon}(t)\,dt, \qquad \forall x \in [-1,1], \tag{5.1}$$

where $\beta_{n,\epsilon}$ is a suitable constant. For $\epsilon > 0$, we can replace (5.1) in (3.6) to recover s_n'' as follows:

$$\begin{aligned} \epsilon^2 s_n''(x) &= \epsilon s_n'(x) - \epsilon\alpha_{n,\epsilon} Q_{n,\epsilon}(x) \\ &= s_n(x) - \beta_{n,\epsilon} - \alpha_{n,\epsilon}\int_{-1}^{x} Q_{n,\epsilon}(t)\,dt - \epsilon\alpha_{n,\epsilon} Q_{n,\epsilon}(x), \qquad \forall x \in [-1,1]. \end{aligned} \tag{5.2}$$

On the other hand, taking the k^{th} derivative of the equation in (3.6) yields

$$-\epsilon s_n^{(k+2)}(x) + s_n^{(k+1)}(x) = \alpha_{n,\epsilon}\, Q_{n,\epsilon}^{(k)}(x), \qquad \forall x \in [-1,1]. \tag{5.3}$$

By induction, (5.2) and (5.3) allow us to write

$$\epsilon^{k+2}\, s_n^{(k+2)}(x) = s_n(x) - \beta_{n,\epsilon} - \alpha_{n,\epsilon} \int_{-1}^{x} Q_{n,\epsilon}(t)\, dt - \alpha_{n,\epsilon} \sum_{m=0}^{k} \epsilon^{m+1}\, Q_{n,\epsilon}^{(m)}(x), \qquad \forall x \in [-1,1], \tag{5.4}$$

where $Q_{n,\epsilon}^{(0)} = Q_{n,\epsilon}$. Since $s_n \in \mathbf{P}_n$ we have $s_n^{(n+1)} = 0$. Thus, with $k = n-1$ in (5.4) one gets

$$s_n(x) = \beta_{n,\epsilon} + \alpha_{n,\epsilon} \left(\int_{-1}^{x} Q_{n,\epsilon}(t)\, dt + \sum_{m=0}^{n-1} \epsilon^{m+1}\, Q_{n,\epsilon}^{(m)}(x) \right), \qquad \forall x \in [-1,1]. \tag{5.5}$$

Now, we impose the boundary conditions obtaining

$$s_n(-1) = 0 = \beta_{n,\epsilon} + \alpha_{n,\epsilon} \sum_{m=0}^{n-1} \epsilon^{m+1}\, Q_{n,\epsilon}^{(m)}(-1), \tag{5.6}$$

$$s_n(1) = 1 = \beta_{n,\epsilon} + \alpha_{n,\epsilon} \left(\int_{-1}^{1} Q_{n,\epsilon}(t)\, dt + \sum_{m=0}^{n-1} \epsilon^{m+1}\, Q_{n,\epsilon}^{(m)}(1) \right). \tag{5.7}$$

Therefore, we finally deduce

$$\alpha_{n,\epsilon} = \left[\int_{-1}^{1} Q_{n,\epsilon}(t)\, dt + \sum_{m=0}^{n-1} \epsilon^{m+1} \left[Q_{n,\epsilon}^{(m)}(1) - Q_{n,\epsilon}^{(m)}(-1) \right] \right]^{-1}. \tag{5.8}$$

At this point we note that problem (2.3) can be also stated as

$$\begin{aligned} -\epsilon\, q_n''(x) + q_n'(x) &= \alpha_{n,\infty}\, Q_{n,\infty}(x), \qquad \forall x \in\,]-1,1[, \\ q_n(-1) &= 0, \\ q_n(1) &= 1, \end{aligned} \tag{5.9}$$

where $Q_{n,\infty}$ is defined in (3.4) and, with the same technique used to obtain (5.8), we have

$$\begin{aligned} \alpha_{n,\infty} &= \left[\int_{-1}^{1} Q_{n,\infty}(t)\, dt + \sum_{m=0}^{n-1} \epsilon^{m+1} \left[Q_{n,\infty}^{(m)}(1) - Q_{n,\infty}^{(m)}(-1) \right] \right]^{-1} \\ &= \begin{cases} \left[\dfrac{2}{n} \sum_{\substack{m=0 \\ m \text{ even}}}^{n-2} \epsilon^{m+1} P_n^{(m+1)}(1) \right]^{-1}, & \text{if } n \text{ is even}, \\ \left[\dfrac{2}{n} \left(1 + \sum_{\substack{m=1 \\ m \text{ odd}}}^{n-2} \epsilon^{m+1} P_n^{(m+1)}(1) \right) \right]^{-1}, & \text{if } n \text{ is odd}. \end{cases} \end{aligned} \tag{5.10}$$

For n fixed, we soon realize that, if ϵ tends to zero and n is even, the polynomial q_n blows up. This is not true for s_n. Actually, for $\epsilon \to 0$, $Q_{n,\epsilon}$ tends uniformly to $Q_{n,0} = P_n/(x - \xi_n^{(n)})$, and by virtue of the Gauss integration formula we have

$$\int_{-1}^{1} Q_{n,0}(t)\, dt = \sum_{k=1}^{n} Q_{n,0}\left(\xi_k^{(n)}\right) w_k = \lim_{x \to \xi_n^{(n)}} \frac{P_n(x)}{x - \xi_n^{(n)}}\, w_n = P_n'\left(\xi_n^{(n)}\right) w_n \neq 0, \tag{5.11}$$

where w_k, $1 \le k \le n$, are the weights of the formula. Hence, $\alpha_{n,\epsilon}$ in (5.8) tends to a finite limit and s_n does not blow up.

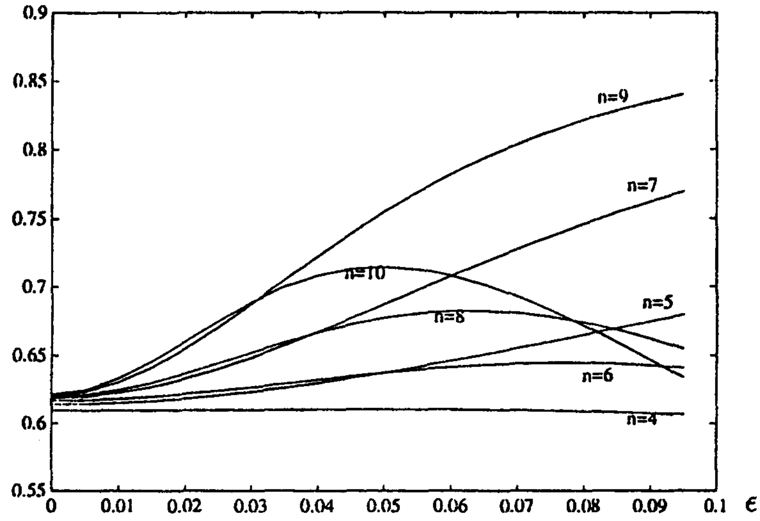

Figure 5. Behavior of $\alpha_{n,\epsilon}$ for $\epsilon \in [0, 1/10]$ and $4 \le n \le 10$.

Thus, by examining (3.6) and (5.9), we realize that the behavior of the corresponding solutions s_n and q_n is strictly related to the magnitude of $\alpha_{n,\epsilon}$. In Figure 5, we plot $\alpha_{n,\epsilon}$ for $\epsilon \in [0, 1/10]$ and various n. The situation for $\alpha_{n,\infty}$ is totally different. According to (5.10), when $n \ll 1/\sqrt{\epsilon}$ one has: $\alpha_{n,\infty} \approx 1/(\epsilon n)$ if n is even, $\alpha_{n,\infty} \approx n/2$ if n is odd. Instead, when ϵ is fixed and $n > 1/\sqrt{\epsilon}$, due to (3.5) $Q_{n,\epsilon}$ approaches $Q_{n,\infty}$ and the two polynomials s_n and q_n are almost coincident with the exact solution u of (2.1).

We conclude this section with a remark. Since the Legendre polynomial P_n satisfies $(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$ we have that (3.1) also takes the form

$$-\epsilon r_n''\left(\tau_j^{(n)}\right) + r_n'\left(\tau_j^{(n)}\right) = 0, \qquad 1 \le j \le n, \tag{5.12}$$

where $r_n = (1 - x^2)P_n' \in \mathbf{P}_{n+1}$ is the polynomial vanishing at the nodes $\eta_j^{(n)}$, $0 \le j \le n$. Therefore, the nodes $\tau_j^{(n)}$, $1 \le j \le n$, can be defined as the zeroes of the polynomial obtained by applying the differential operator $-\epsilon \frac{d^2}{dx^2} + \frac{d}{dx}$ to the oscillating function r_n. Since $r_n(\pm 1) = 0$, we deduce that the polynomial $s_n + \gamma r_n \in \mathbf{P}_{n+1}$ satisfies the collocation problem (3.3) for any $\gamma \in \mathbf{R}$. This is not true with other distributions of nodes. Hence, the new grid is the one providing an approximated polynomial $s_n \in \mathbf{P}_n$ (corresponding to the case $\gamma = 0$) that would be also solution in the larger space $\mathbf{P}_{n+1}$. This fact could also explain why the new technique performs better than the standard one.

6. EXTENSIONS

The case of the differential equation

$$-\epsilon u''(x) + a(x)u'(x) = f(x), \qquad x \in]-1, 1[, \tag{6.1}$$

can be treated by defining the $\tau_j^{(n)}$ to be the zeroes of

$$-\epsilon P_n'\left(\tau_j^{(n)}\right) + a\left(\eta_j^{(n)}\right) P\left(\tau_j^{(n)}\right) = 0. \tag{6.2}$$

The location of $\tau_j^{(n)}$ with respect to $\eta_j^{(n)}$ depends on the sign of $a(\eta_j^{(n)})$. In fact the generalization of (3.2) is

$$\tau_j^{(n)} \in \begin{cases}]\xi_j^{(n)}, \eta_j^{(n)}[, & \text{if } a\left(\eta_j^{(n)}\right) > 0, \\ ]\eta_j^{(n)}, \xi_{j+1}^{(n)}[, & \text{if } a\left(\eta_j^{(n)}\right) < 0. \end{cases} \tag{6.3}$$

When $a(\eta_j^{(n)}) = 0$, which means that locally the differential operator is only diffusive, one has $\tau_j^{(n)} = \eta_j^{(n)}$, reobtaining the usual Legendre node.

In the case of more dimensions, the partial differential equation

$$-\epsilon\Delta u + \vec{\beta} \cdot \vec{\nabla} u = f, \qquad \text{in }]-1,1[\times]-1,1[, \tag{6.4}$$

provided of some boundary conditions, can be approximated by a collocation method where any node of the grid is a perturbation of the point $(\eta_j^{(n)}, \eta_i^{(n)})$ in the direction opposite to the flux $\vec{\beta}$ and at a distance depending on the ratio $(1/\epsilon)|\vec{\beta}|$. The details of the implementation are discussed in [5]. Experiments have been also carried out with success in the approximation of nonlinear time-dependent problems.

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